MGSC 1205

Quantitative Methods I

Slides Three – LP I: Introduction & Graphical Solution Method

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Linear Programming – Introduction

- Management decisions in many organizations involve trying to make most effective use of ***resources***.
  - Machinery, labor, money, time, warehouse space, and raw materials. These resources can be used to:

  →Produce products – such as computers, automobiles, or clothing, … etc.

  →Provide services – such as package delivery, health services, or investment decisions.
- To solve problems of *resource allocation* one may use *mathematical programming*.

Mathematical Programming & *LP*

- *Programming* refers to setting up and solving a problem mathematically.
- Linear programming (*LP*) is most common type of *mathematical programming*.
- One assumes all relevant input data and parameters are known with certainty in models (*deterministic* models).
- Computers play important in *LP*.

Properties of a *LP* Model

- All problems seek to *maximize* or *minimize* some quantity, usually profit or cost (called ***objective function***).
- LP models usually include restrictions, or ***constraints***, limit degree to which one can pursue our objective. LP model usually includes a set of ***non-negativity constraints.***
- There must be alternative courses of action from which to choose.
- Objective and constraints in *LP* problems must be expressed in terms of ***linear equations*** or ***inequalities***.

Linear Equations and Inequalities

- This is a linear equation:

$$2A + 5B = 10$$

- This equation is not linear:

$$2A^2 + 5B^3 + 3AB = 10$$

- LP uses, in many cases, inequalities like:

$$A + B \leq C \quad \text{or} \quad A + B \geq C$$

An inequality has a $\leq$ or $\geq$ sign.

Three Steps of Developing LP Problem

Formulation.

- Process of translating problem scenario into LP model framework with set of mathematical relationships.

Solution.

- Mathematical relationships resulting from the formulation process are solved to identify an optimal solution.

Interpretation and Sensitivity Analysis.

- Problem solver or analyst works with manager to:
  - Interpret results and implications of problem solution.
  - Investigate changes in input parameters and model variables and impact on problem solution results.

Formulating a LP Problem

- A common LP application is a product mix problem.
  - Two or more products are usually produced using limited resources – such as personnel, machines, raw materials, and so on.
- Profit firm seeks to maximize is based on profit contribution per unit of each product.
- Firm would like to determine
  - How many units of each product it should produce.
  - Maximize overall profit given its limited resources.

Example: Flair Furniture Company

Company Data and Constraints:

- Flair Furniture Company produces tables and chairs.
- Each table sold results in $7 profit, while each chair produced yields $5 profit.
- Each table requires: 4 hours of carpentry and 2 hours of painting.
- Each chair requires: 3 hours of carpentry and 1 hour of painting.
- Available production capacity: 240 hours of carpentry time and 100 hours of painting time.
- Due to existing inventory of chairs, Flair is to make no more than 60 new chairs.

Flair Furniture's problem:

- Determine best possible combination of tables and chairs to manufacture in order to attain maximum profit.

Decision Variables

- The *decision variables* are the elements under control of the model developer and their values determine the solution of the model.
- Problem facing Flair is to determine how many chairs and tables to produce to yield maximum profit?
- In Flair Furniture problem, there are ***two*** ***unknown variables*****.**

Objective Function

- Objective function states goal of problem
  - What major objective is to be solved?
  - Maximize profit!
- A LP model *must* have a ***single*** objective function.

In Flair's problem, total profit may be expressed as:

Profit = ($7 profit per table) × T + ($5 profit per chair) × C

= 7 T + 5 C,

where,

T is # tables produced and *C* is the # chairs produced.

T and C are the decision variables.

Constraints

- Denote conditions that prevent one from selecting any specific subjective value for decision variables.

  - There are 240 carpentry hours available.
  - There are 100 painting hours available.
  - The marketing specified chairs limit constraint.
  - The ***non-negativity*** constraints.

Constraints

- In Flair Furniture's problem, there are ***four restrictions*** on solution:
  - Restrictions 1 and 2 have to do with available carpentry and painting times, respectively.
  - Restriction 3 is concerned with upper limit on number of chairs.
  - A sign restriction is associated with each variable. For any variable, the sign restriction specifies that the variable must be non-negative.

Basic Assumptions of a LP Model

- Conditions of *certainty* exist.
- *Proportionality* in objective function and constraints (1 unit → 3 hours, 3 units → 9 hours).
- *Additively* (total of all activities equals sum of individual activities).
- *Divisibility* assumption that solutions need not necessarily be in whole numbers (integers).

The completed formulation

- Maximize: $P = 7\,T + 5\,C$
- Subject to (constraints)

➢ Carpentry restriction: $4\,T + 3\,C \leq 240$

➢ Painting restriction: $2\,T + C \leq 100$

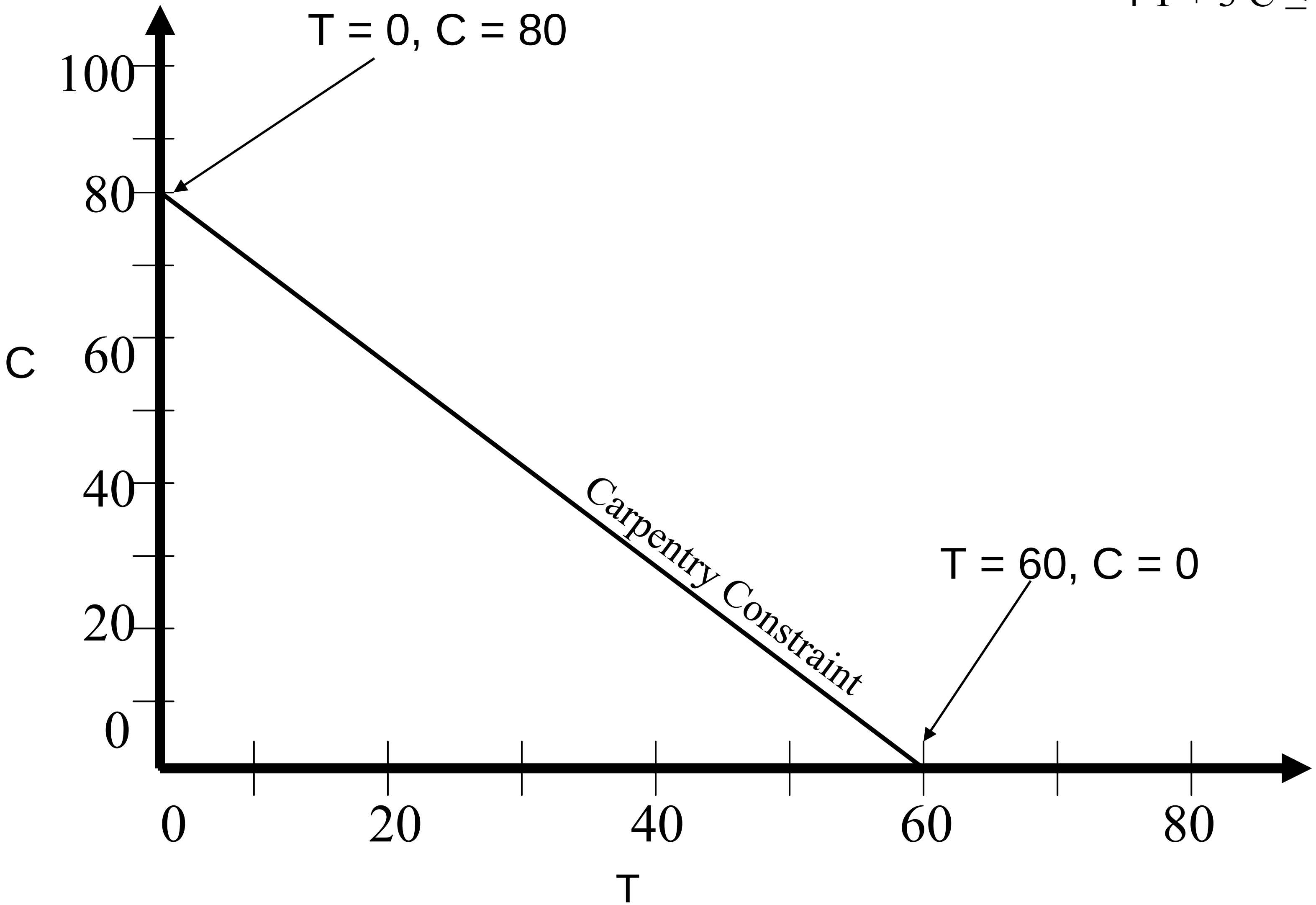
➢ Inventory condition: $C \leq 60$

➢ Non-negativity condition: $T \geq 0,\ C \geq 0$

Graphical Solution

1) Carpentry Constraint

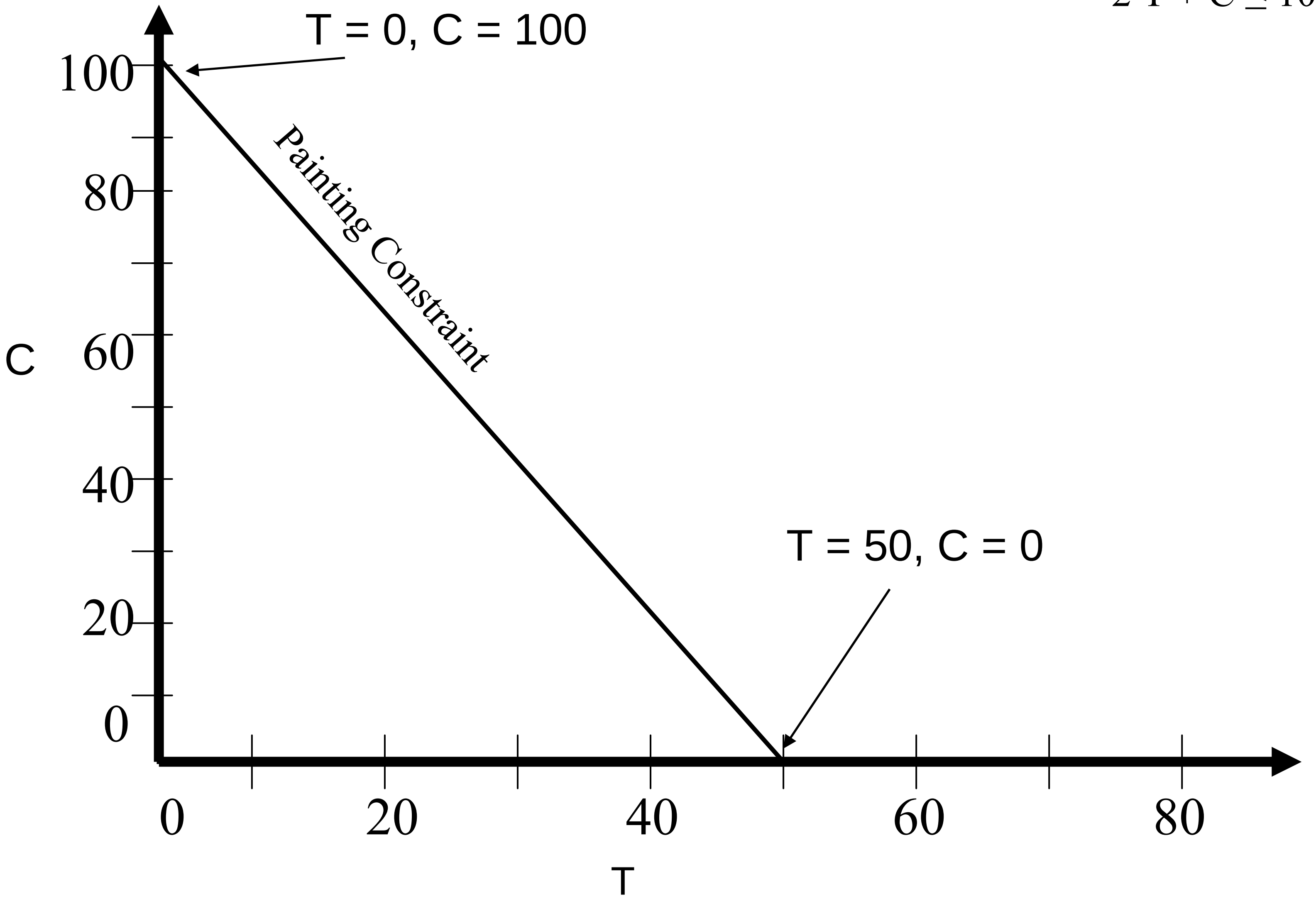
$4\,T + 3\,C \leq 240$



Graphical Solution

2) Painting Constraint

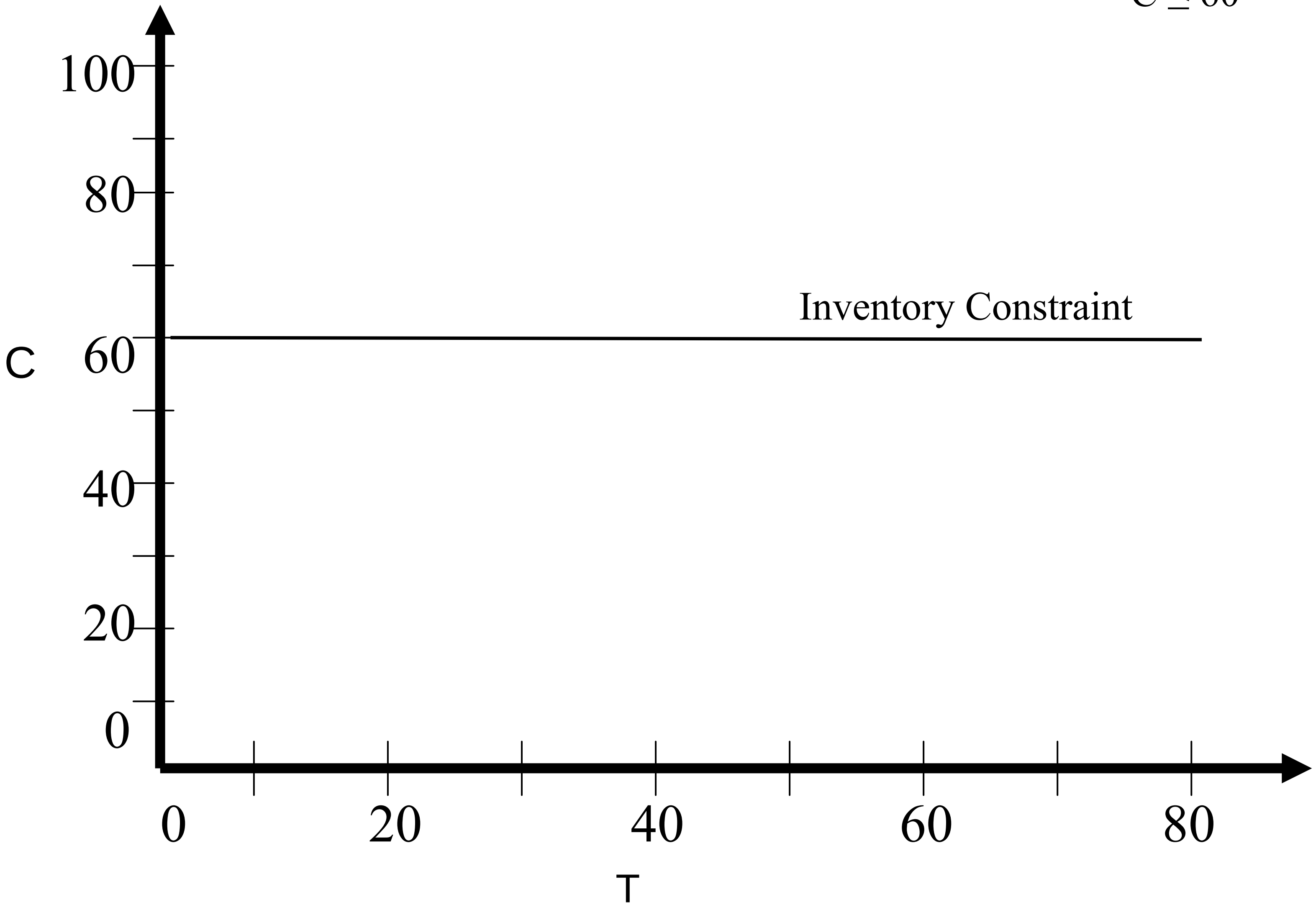
$2\,T + C \leq 100$



Graphical Solution

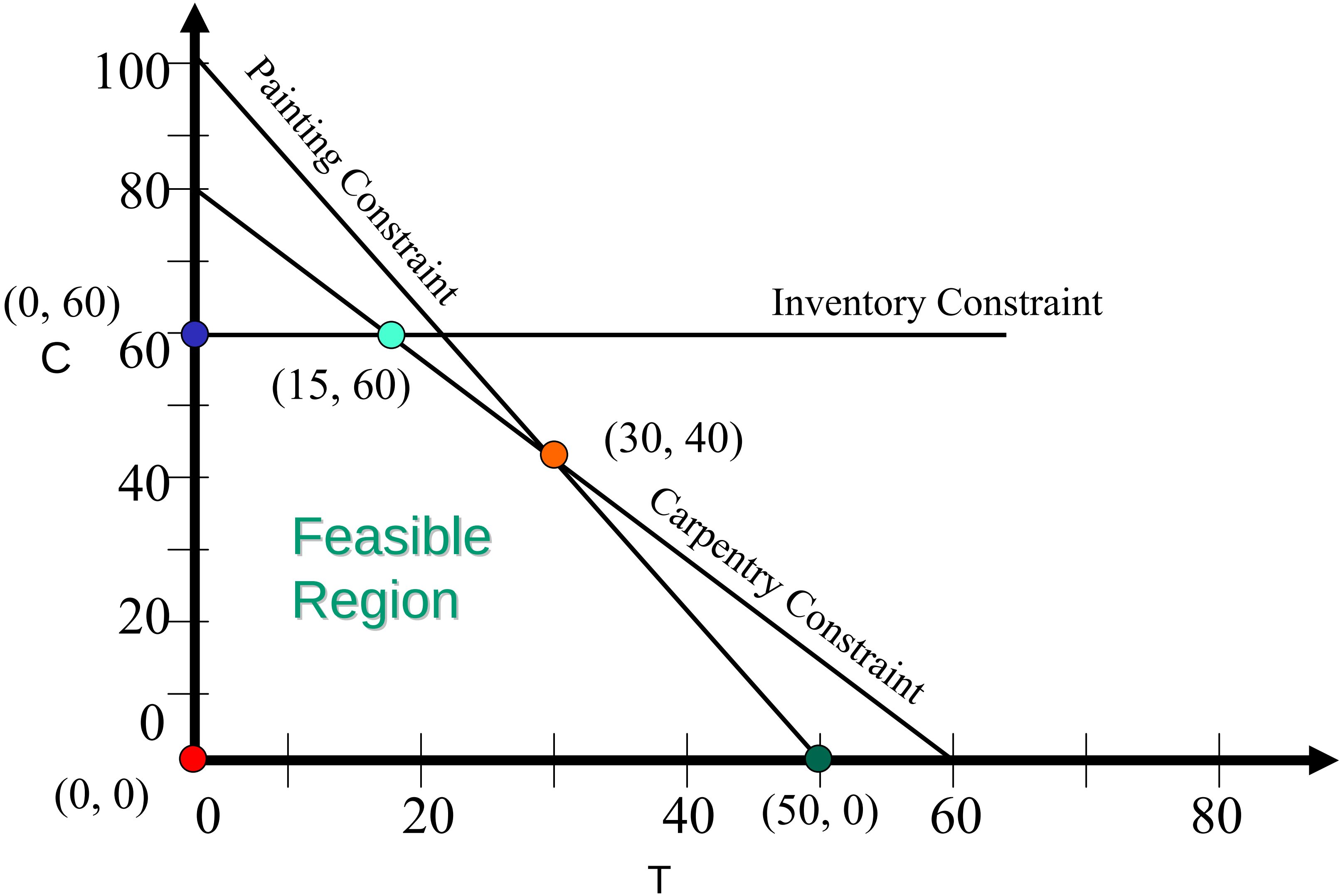
3) Inventory Constraint

$C \leq 60$

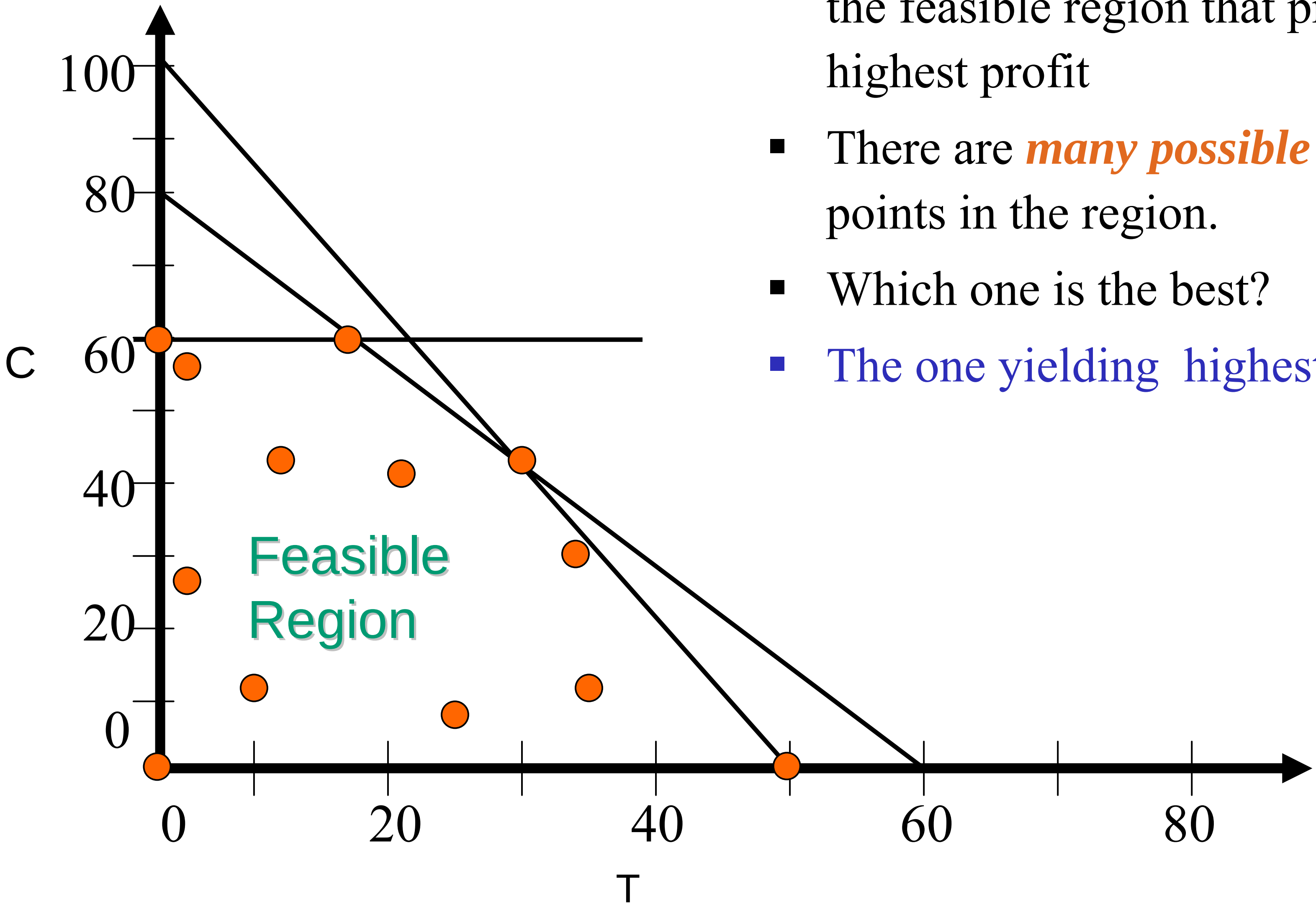


Graphical Solution

Feasible Region – Satisfies all constraints



Graphical Solution

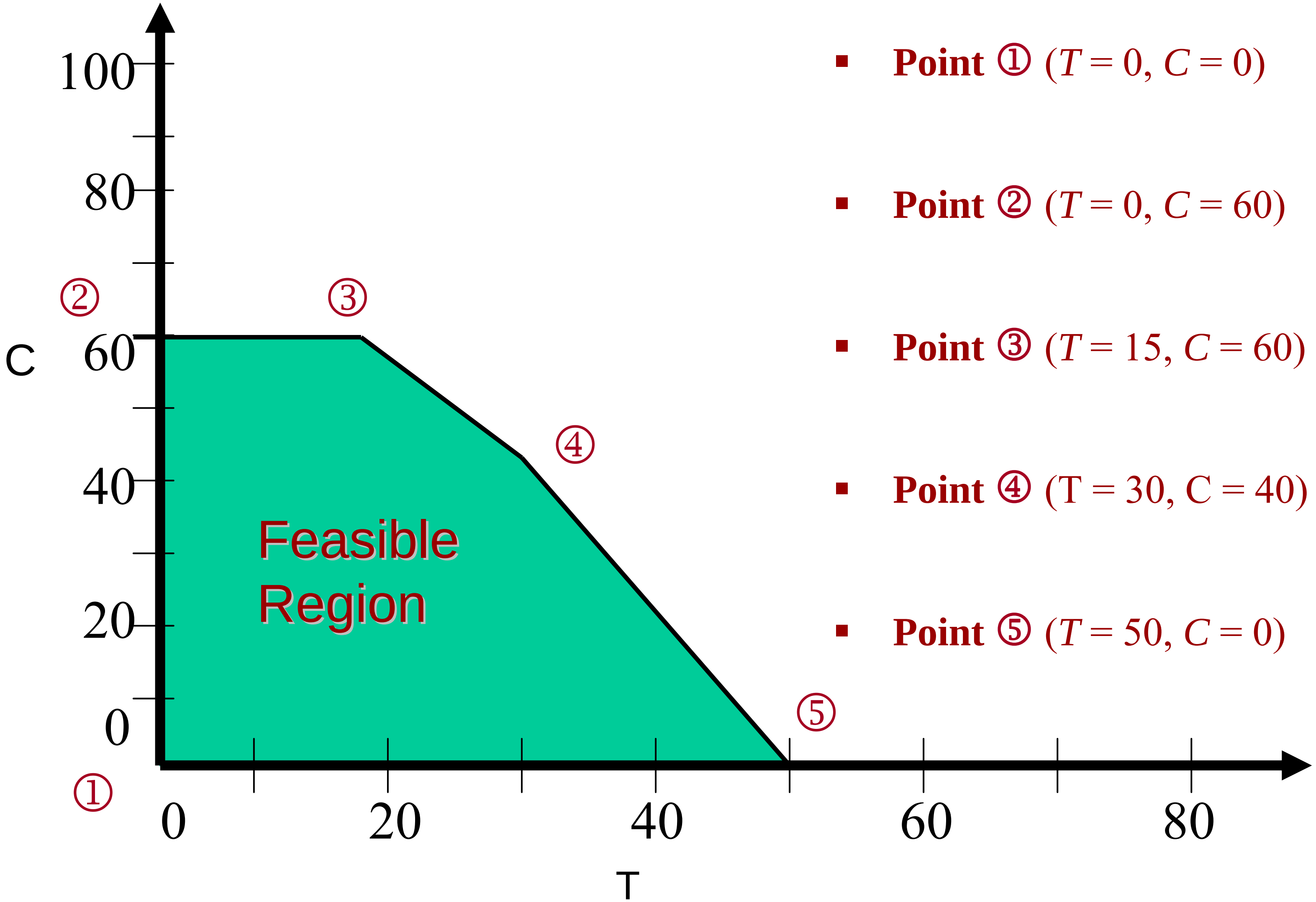


- Optimal solution is the point in the feasible region that produces highest profit
- There are ***many possible*** solution points in the region.
- Which one is the best?
- The one yielding highest profit?

Graphical Solution

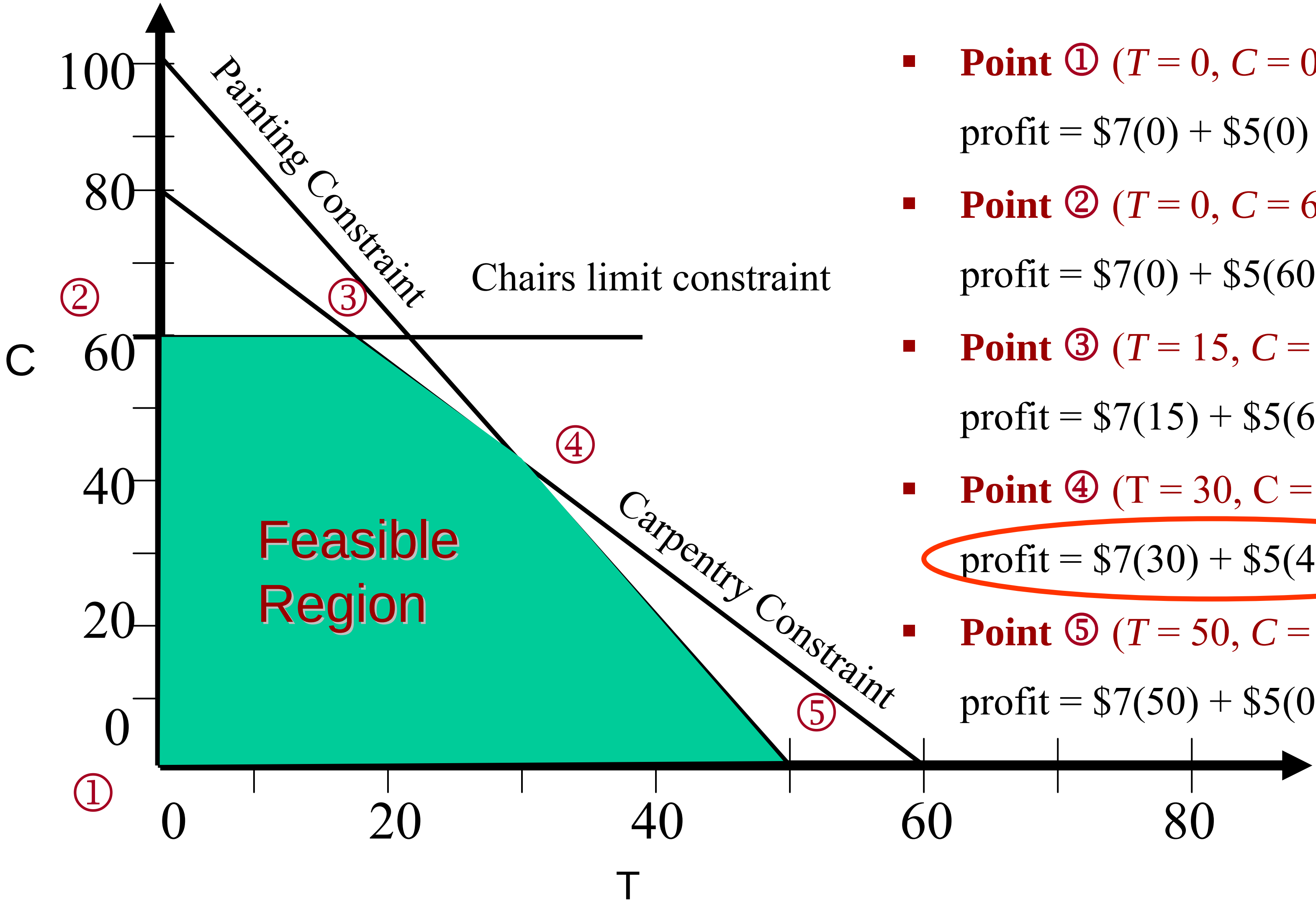
- We can prove that ***the optimal solution*** always exists at the intersection of constraints (corner points).
  - Known as **Corner Point Property.**
- Why not just go directly to the places where the corner points?

Corner Point Solution Method



- **Point ①** ($T = 0$, $C = 0$)
- **Point ②** ($T = 0$, $C = 60$)
- **Point ③** ($T = 15$, $C = 60$)
- **Point ④** (T = 30, C = 40)
- **Point ⑤** ($T = 50$, $C = 0$)

Graphical Solution Procedure



- **Point ①** ($T = 0$, $C = 0$)

  profit = \$7(0) + \$5(0) = \$0
- **Point ②** ($T = 0$, $C = 60$)

  profit = \$7(0) + \$5(60) = \$300
- **Point ③** ($T = 15$, $C = 60$)

  profit = \$7(15) + \$5(60) = \$405
- **Point ④** (T = 30, C = 40)

  profit = \$7(30) + \$5(40) = \$410
- **Point ⑤** ($T = 50$, $C = 0$)

  profit = \$7(50) + \$5(0) = \$350

Example 2

DVD	2 hours electronics work 4 hours assembly time
MP3	3 hours electronics work 1 hours assembly time

- Hours available: 3000 (elect), 2000 (assembly time)
- Profit / unit: DVD \$8, MP3 \$10

D = number of DVD players to make

M = number of mp3 players to make

Example

DVD	2 hours electronics work 1 hours assembly time
MP3	3 hours electronics work 2 hours assembly time

- Hours available: 3000 (elect) 2500 (assembly time)
- Profit / unit: DVD $8, MP3 $10

D = number of DVD players to make

M = number of mp3 players to make

Example: Holiday Meal Turkey Ranch

- Buy two brands of feed for good, low-cost diet for turkeys.
- Each feed may contain three nutritional ingredients (protein, vitamin, and iron).

✓ One pound of **Brand A** contains:
- ✓ 5 ounces of protein,
- ✓ 4 ounces of vitamin, and
- ✓ 0.5 ounces of iron.

✓ One pound of **Brand B** contains:
- ✓ 10 ounces of protein,
- ✓ 3 ounces of vitamins, and
- ✓ 0 ounces of iron.

✓ **Minimum monthly requirement per turkey (OZ)**:
Protein: 90, Vitamin: 48, Iron: 1.5

✓ **Brand A** feed costs ranch **$0.02** per pound, while **Brand B** feed costs **$0.03** per pound.

- Ranch owner would like lowest-cost diet that **meets minimum monthly intake requirements** for each nutritional ingredient.

Example - Fall 2009

A pension fund manager decides to invest at most \$40 million in bonds, paying 12% annual interest, and in mutual funds, paying 8% annual interest. He plans to invest at least \$20 million in bonds and at least \$15 million in mutual funds. Bonds have an initial fee of \$300 per million dollars invested, while the fee for mutual funds is \$100 per million. The fund manager is allowed to spend no more than \$8400 on fees. How much should be invested in each to maximize annual interest? Formulate an LP model for the problem.

Conditions	Bonds	Mutual Funds	Resources Available
Total funds (million)	1	1	\$40 million
Lower limit on Bonds	At least \$20 m		
Lower limit on Mutual Funds		At least \$15 m	
Initial fee	\$300	\$100	\$8,400
Interest	12%	8%	maximize

x = amount invested in bonds (in million dollars); y = amount invested in mutual funds (in million dollars).

Maximize $Interest = 0.12x + 0.08y$

Subject to: $x + y \le 40$ total amount invested

$x \ge 20$ lower limit for bonds

$y \ge 15$ lower limit for mutual funds

$300x + 100y \le 8{,}400$ initial fees

$x \ge 0, \quad y \ge 0$

Example - Fall 2009

Irwin Textile Mills produces two types of cotton cloth: denim and corduroy. Corduroy is a heavier grade of cotton cloth and, as such, requires 3.75 kilograms of raw cotton per metre, whereas denim requires 2.5 kilograms of raw cotton per metre. A metre of corduroy requires 3.5 hours of processing time; a metre of denim requires 3 hours. The manufacturer has 3,000 kilograms of raw cotton and 3,000 hours of processing time available each month. The manufacturer makes a profit of $4.5 per metre of denim and $6.25 per metre of corduroy. Due to market considerations, the manufacturer wants to produce at least 300 metres of corduroy each month.

The manufacturer wants to know how many metres of each type of cloth to produce each month to maximize its profit. Formulate an LP model for the problem.

Conditions	corduroy	denim	Resources Available
Material (kilograms)	3.75	2.5	3,000 kg
Processing time (hours)	3.5	3	3,000 hours
Lower limit on corduroy	At least 300 m		
Profit per metre	$6.25	$4.50	maximize

Let x = number of metres of corduroy to produce; y = number of metres of denim to produce.

Maximize $P = 6.25x + 4.5y$

Subject to:

$3.75x + 2.5y \leq 3{,}000$ raw cotton constraint

$3.5x + 3y \leq 3{,}000$ labour hours

$x \geq 300$ lower limit on corduroy

$x \geq 0, \; y \geq 0$